An Unconditionally Stable 3-D ADI-MRTD Method Free of the CFL Stability Condition

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***Abstract*—In this paper, an alternating direction implicit (ADI) technique is applied to the recently developed multiresolution time-domain (MRTD) method, resulting in an unconditionally stable ADI-MRTD scheme free of the Courant–Friedich–Lecy (CFL) stability condition. The unconditional stability is theoretically proved, and preliminary numerical results are presented to validate the scheme. Because the scheme is now free of the stability condition, its time step is determined only by modeling accuracy. The price for having the unconditional stability is, however, that the required computation memory becomes almost twice of that for the original MRTD.**

***Index Terms*—Alternating direct implicit (ADI) technique and unconditionally stability, FDTD, MRTD.**

I. Introduction

FDTD is a powerful numerical method for solving electromagnetic problems where field components are computed in a time recursive fashion. An extensive review of the state-of-the-art was presented in [1] and [2]. Nevertheless, FDTD is very computationally intensive due to its two inherent physical constraints, one being the numerical dispersion and another being the numerical stability. To make the numerical dispersion small, the spatial step of FDTD must be small, normally smaller than one-tenth of wavelength. To make the time-recursion stable, the time step must also be small, smaller than the so-called Courant–Friedich–Lecy (CFL) stability condition. As a result, a large numerical mesh and a long simulation time may be required for solving electrically large structures.

Many efforts have been made in relaxing or removing the above two constraints in order to reduce the computational expenditures. Among them are the multiresolution time-domain (MRTD) method applying wavelets expansions in space [3] and pseudospectral time-domain (PSTD) method applying fast Fourier transform (FFT) in space [4]. Both methods have claimed to achieve low numerical dispersion with numerical grid resolutions as low as two points per wavelength. However, the CFL stability condition for both methods still remains.

To relax or even remove the CFL conditions, implicit methods may be incorporated in formulating the FDTD. One of the implicit approaches is the alternating direction implicit (ADI) method that has been applied in heat transfer problems [5]. An ADI-FDTD was attempted in 1980 [6] but was not successful until recently [7]–[9]. The ADI was applied to a two-dimensional (2-D) TE-wave in [7]. A three-dimensional (3-D) full-wave, unconditionally stable ADI-FDTD was shown in [8], with rigorous proof of unconditional stability, and in [9], with excellent numerical examples (in particular for the low-frequency computations). In this paper, further to our work in [8], the ADI principle is applied to the MRTD, resulting in an unconditionally stable 3-D ADI-MRTD algorithm.

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II. Unconditionally Stable 3-D ADI-MRTD Formulation

Similar to the development of the ADI-FDTD described in [8], the ADI-MRTD is derived by applying the ADI algorithm to the original MRTD formulations. Since various forms of MRTD exist due to the selections of different wavelet functions, various forms of ADI-MRTD can also be developed. In our case, the S-MRTD [3] is considered for simplicity as an illustration of the ADI-MRTD development. The field space is discretized with the Yee's grid, and the field quantities are expanded in terms of the wavelet expansions [3]. For instance, one can have

$$\begin{aligned} E_x|_{i,j,k}^{n} - E_x|_{i,j,k}^{n-1} &= \frac{\Delta t}{\varepsilon \Delta y} \sum_{m=1-M}^{M} a(m) H_z|_{i,j+m,k}^{n-1/2} \\ &\quad - \frac{\Delta t}{\varepsilon \Delta z} \sum_{m=1-M}^{M} a(m) H_y|_{i,j,k+m}^{n-1/2} \qquad (1) \end{aligned}$$

$$\begin{aligned} H_x|_{i,j,k}^{n+1/2} - H_x|_{i,j,k}^{n-1/2} &= \frac{\Delta t}{\mu \Delta z} \sum_{m=-M}^{M-1} c(m) E_y|_{i,j,k+m}^{n} \\ &\quad - \frac{\Delta t}{\mu \Delta y} \sum_{m=-M}^{M-1} c(m) E_z|_{i,j+m,k}^{n} \qquad (2) \end{aligned}$$

where $a(m)$ and $c(m)$ are the coefficients resulted from the wavelet expansions [3].

When the ADI scheme is applied, the above equations are broken up into two sub-steps, nth step and $(n+(1/2))$th step. In the first sub-step, the time instants for all the first terms on the right-hand sides are set to be same as the first terms on the left hand sides . In the second sub-step, the time instants for all the second terms on the right-hand sides are set to be same as the first-terms on the left-hand side. More specifically, (1) and (2) now become

$$\begin{aligned} E_x|_{i,j,k}^{n} - E_x|_{i,j,k}^{n-1/2} &= \frac{\Delta t}{2\varepsilon\Delta y}\sum_{m=1-M}^{M} a(m)H_z|_{i,j+m,k}^{n} \\ &\quad - \frac{\Delta t}{2\varepsilon\Delta z}\sum_{m=1-M}^{M} a(m)H_y|_{i,j,k+m}^{n-1/2} \end{aligned} \tag{3}$$

$$\begin{aligned} H_x|_{i,j,k}^{n} - H_x|_{i,j,k}^{n-1/2} &= \frac{\Delta t}{\mu\Delta z}\sum_{m=-M}^{M-1} c(m)E_y|_{i,j,k+m}^{n} \\ &\quad - \frac{\Delta t}{\mu\Delta y}\sum_{m=-M}^{M-1} c(m)E_z|_{i,j+m,k}^{n-1/2} \end{aligned} \tag{4}$$

for the first substep, and

$$\begin{aligned} E_x|_{i,j,k}^{n+1/2} - E_x|_{i,j,k}^{n} &= \frac{\Delta t}{2\varepsilon\Delta y}\sum_{m=1-M}^{M} a(m)H_z|_{i,j+m,k}^{n} \\ &\quad - \frac{\Delta t}{2\varepsilon\Delta z}\sum_{m=1-M}^{M} a(m)H_y|_{i,j,k+m}^{n+1/2} \end{aligned} \tag{5}$$

$$\begin{aligned} H_x|_{i,j,k}^{n+1/2} &= H_x|_{i,j,k}^{n} + \frac{\Delta t}{\mu\Delta z}\sum_{m=-M}^{M-1} c(m)E_y|_{i,j,k+m}^{n} \\ &\quad - \frac{\Delta t}{\mu\Delta y}\sum_{m=-M}^{M-1} c(m)E_z|_{i,j+m,k}^{n+1/2} \end{aligned} \tag{6}$$

for the second substep.

By applying the same procedure to the MRTD equations for the other components, the complete formulations for the unconditionally stable ADI-MRTD scheme can be obtained.

In comparisons of (4)–(6) for the ADI-MRTD with (1) and (2) for the original MRTD, it is not difficult to see that memory required with the ADI-MRTD is double that of the original MRTD. This is because, in the ADI-MRTD, both electric and magnetic field components are updated at each sub-step, while in the original MRTD, either electric or magnetic field components are updated at each sub-step as shown in (1) and (2).

The ADI-MRTD formulations can be further simplified as presented in [8] for the ADI-MRTD. For instance, after some tedious derivations, one can have following equations for E_x: for the first substep

$$\begin{aligned} E_x|_{i,j,k}^{n} &- \frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\sum_{m=1-M}^{M} a(m)\sum_{l=-L}^{L-1} c(l)E_x|_{i,j+l+m,k}^{n} \\ &= E_x|_{i,j,k}^{n-1/2} + \frac{\Delta t}{2\varepsilon\Delta y}\sum_{m=1-M}^{M} a(m)H_z|_{i,j+m,k}^{n-1/2} \\ &\quad - \frac{\Delta t}{2\varepsilon\Delta z}\sum_{m=1-M}^{M} a(m)H_y|_{i,j,k+m}^{n-1/2} \\ &\quad - \frac{\Delta t^2}{4\varepsilon\mu\Delta x\Delta y}\sum_{m=1-M}^{M} a(m)\sum_{l=-L}^{L-1} c(l)E_y|_{i+l,j+m,k}^{n-1/2} \end{aligned} \tag{7}$$

while for the second substep

$$\begin{aligned} E_x|_{i,j,k}^{n+1/2} &- \frac{\Delta t^2}{4\varepsilon\mu\Delta z^2}\sum_{m-1-M}^{M} a(m)\sum_{l=-L}^{L-1} c(l)E_x|_{i,j,k+l+m}^{n+1/2} \\ &= E_x|_{i,j,k}^{n} + \frac{\Delta t}{2\varepsilon\Delta y}\sum_{m=1-M}^{M} a(m)H_z|_{i,j+m,k}^{n} \\ &\quad - \frac{\Delta t}{2\varepsilon\Delta z}\sum_{m=1-M}^{M} a(m)H_y|_{i,j,k+m}^{n} \\ &\quad - \frac{\Delta t^2}{4\mu\varepsilon\Delta x\Delta z}\sum_{m-1-M}^{M} a(m)\sum_{l=-L}^{L-1} c(l)E_z|_{i+l,j,k+m}^{n}. \end{aligned} \tag{8}$$

The advantages of solving (7) and (8) instead of (3) and (5) lies in the fact that, in (7) and (8), the quantities to be updated are of the same quantity, E_xs, on the left-hand side of the equations. They are neighboring to each other along only *one* dimension, the k direction (representing the spatial direction in the z direction). By changing the subscript k, (5) or (6) each leads to a linear system of equations with a banded coefficient matrix whose width is determined by the order of MRTD applied. Such a banded matrix can be solved quickly for the MRTD time recursions with special mathematic subroutines.

III. Numerical Stability

The unconditional stability of the proposed ADI-MRTD can be proven in a way similar to that shown in [8]. By taking the discrete Fourier transform in the spatial domain along the three directions respectively, the ADI-MRTD in the spatial spectral domain at the nth step can be written as

$$\boldsymbol{F}^{n} = M\boldsymbol{F}^{n+1} \tag{9}$$

where $\boldsymbol{F} = [E_x, E_y, E_z, H_x, H_y, H_z]^T$ is the six field components in the spectral domain. M is a 6×6 matrix whose coefficients are functions of space steps, time steps, medium constitutive parameters, and spatial frequencies in the x, y, and z directions respectively.

For the proposed ADI-MRTD to be unconditionally stable, magnitudes of all the eigenvalues of matrix M need to be either unity or smaller. After some tedious derivations with the help of MAPLE5.1, they are found to be

$$\begin{aligned} \lambda_1 &= \lambda_2 = 1 \\ \lambda_3 &= \lambda_4 = \frac{T + jS}{R} \\ \lambda_3 &= \lambda_4 = \frac{T - jS}{R} \end{aligned} \tag{10}$$

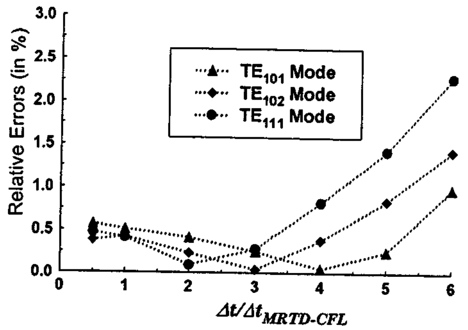

Fig. 1. Relative errors of the ADI-MRTD with the dense mesh ($\Delta t_{\mathrm{MRTD\text{-}CFL}}$ is the CFL time step limit of the original MRTD).

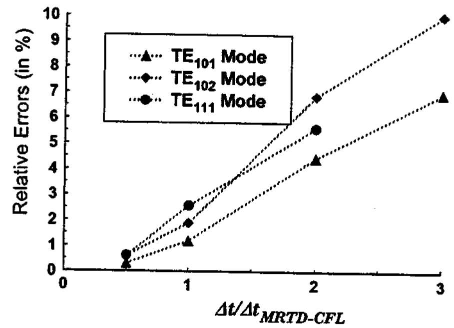

Fig. 2. Relative errors of the ADI-MRTD with the coarse mesh ($\Delta t_{\mathrm{MRTD\text{-}CFL}}$ is the CFL time step limit of the original MRTD).

where

$$R = \left(\mu\varepsilon + W_x^2\right)\left(\mu\varepsilon + W_y^2\right)\left(\mu\varepsilon + W_z^2\right) \tag{11}$$

$$S = \sqrt{\begin{aligned}&4\mu\varepsilon\big(\mu\varepsilon W_x^2 + \mu\varepsilon W_y^2 + \mu\varepsilon W_z^2 + W_x^2W_y^2 + W_y^2W_z^2\\ &+ W_z^2W_z^2\big)\left(\mu^3\varepsilon^3 + W_x^2W_y^2W_z^2\right)\end{aligned}} \tag{12}$$

$$\begin{aligned}T = \mu^3\varepsilon^3 - \mu\varepsilon\,\big(&\mu\varepsilon W_x^2 + \mu\varepsilon W_y^2 + \mu\varepsilon W_z^2 + W_x^2W_y^2\\ &+ W_y^2W_z^2 + W_z^2W_z^2\big) + W_x^2W_y^2W_z^2\end{aligned} \tag{13}$$

$$W_\alpha = \frac{\Delta t}{\Delta \alpha}\sum_{m=1}^{M} a(m)\sin\left(\frac{(2m-1)k_\alpha\Delta\alpha}{2}\right), \quad \alpha = x,\, y,\, z. \tag{14}$$

As shown in [8]

$$R^2 = S^2 + T^2. \tag{15}$$

As a result, magnitudes of all the eigenvalues, as represented by (10), are of unity. Therefore, the proposed ADI-MRTD scheme is unconditionally stable.

IV. Numerical Results

In order to validate the proposed ADI-MRTD, a cavity with dimensions of 90 mm × 60 mm × 150 mm was computed. In the first case, the mesh chosen was rather dense. The spatial steps in all the three dimensions were chosen to be $\Delta l = 5$ mm, leading to a 18 × 12 × 30 grid. The time step was varied from one to six times of the CFL limit of the original S-MRTD [3]. Fig. 1 shows the relative errors from the analytical solutions for the three resonant modes, TE_{101}, TE_{102}, and TE_{103}. The horizontal axis is the ratio of the ADI-MRTD time step Δt to the CFL time step limit of the original MRTD $\Delta t_{\mathrm{MRTD\text{-}CFL}}(=0.368\,112(\Delta l/c))$. It can be seen that, when time step increases to six times of CFL of MRTD, the errors become bigger but are still relatively small, less than 3%. In other words, the number of iteration in this case can be saved by up to six times compared with the original MRTD.

In the second case, a coarse mesh was chosen with the space step being 30 mm. It led to a 3 × 2 × 5 grid. Fig. 2 shows the computation results. As can be seen now, with a coarse mesh, the time step can no longer be too large. Even with the time step of twice the CFL limit of the original MRTD, the error with the ADI-MRTD becomes larger than 4.4%. Further studies show that such increases in errors are mainly due to the larger numerical dispersion errors because the absolute value of the time step Δt with the coarse mesh is larger than that with the dense mesh. For instance, for the resonant frequency of $f = 2.06$ GHz, $f\Delta t = 0.0248$ with the dense mesh and $f\Delta t = 0.149$ with the coarse mesh when $\Delta t = 2\Delta t_{\mathrm{MRTD\text{-}CFL}}$ is chosen. As a result, we conclude that the time step with the unconditionally stable ADI-MRTD is now mainly constrained by the modeling accuracy.

V. Conclusion

In this paper, the unconditionally stable ADI-MRTD scheme was developed. The CFL stability condition is completely removed, and the time step is now only determined by the modeling accuracy required. The ADI-MRTD scheme is particularly useful with the variable mesh schemes where the time step can be taken uniformly the same regardless of the mesh size in a numerical grid.

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